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The founding of Italian vernacular algebra

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Jacopo or ps.-Jacopo?

In [1929], Louis Karpinski published an article on "The Italian Arithmetic and Algebra of Master Jacob of Florence, 1307". Here, he described the Vatican manuscript Lat. 4826 (henceforth **V**), while pointing out that another manuscript of Jacopo's work, Riccardiana Ms. No 2236 (henceforth **F**), had been mentioned by Boncompagni in 1883, and discussed by the librarian of the Riccardiana in 1754. He had not seen this version of the treatise, and his sources did not allow him to discover the differences between the two manuscripts - in particular the absence of the algebra section from **F**.

What he did discover were some of the differences between the algebra of **V** and earlier Latin writings on the subject - the translations of al-Khwārizmī and Abū Kāmil as well as the *Liber abbaci*. He pointed out that both the examples and the order of the cases deviate from the earlier models; that one of the "new" examples is already found with Mahāvīra though with other numbers;[1] he also mentioned the presence of rules for solving 14 reducible third- and fourth-degree equations, but did not tell explicitly that **V** gives no geometrical proofs for its solutions of the mixed second-degree problems.

1929 fell in the middle of a period where the interest in European medieval mathematics was at its lowest ebb since 1840 (perhaps since the very Middle Ages); moreover, few pieces of *abbaco* mathematics were known by historians of mathematics,[2]1 - so few indeed that they were not recognized to constitute a mathematical genre of their own. There may therefore be little reason to wonder that Karpinski's observations went unnoticed.

In [1976: 488*f*, 517], Warren Van Egmond described both manuscripts, referring also to Karpinski's description of **V**. In [1980: 166*f*], he described

1 He did not mention, however, that Mahāvīra's method [trans. Raṅgācārya 1912: 151] is quite different (and much more straightforward), which diminishes the significance of the observation.

2 [Libri 1838: III, 302?356] contains extracts from Piero della Francesca's *Trattato d'abaco* and from the *Aliabraa-Argibra*; Karpinski [1910] had described a third treatise, and Elisabeth Buchanan Cowley a fourth in a publication from 1923 which I have not seen, but which is mentioned in the preface to the edition of the text in [Vogel 1977]. I may have overlooked a few other items, but probably not much with which historians of mathematics were familiar at the epoch.

a third manuscript, Trivulziana No. 90 (Milan; henceforth **M**; written in Genova around 1410), whose contents is similar to **F**.

None the less, [Van Egmond 1978] identifies Paolo Gherardi's *Libro di ragioni* from 1328 as "the earliest vernacular treatment of algebra" without referring to Jacopo's treatise; indeed, as Van Egmond tells me in a personal communication, the fact that **V** was written in the mid-fifteenth century and contains rules for the fourth degree not present in Gherardi's treatise makes him conclude "that the algebra section of Vat.Lat. 4826 is a late 14th-century algebra text that has been inserted into a copy of Jacopo's early 14th-century algorism by a mid-15th-century copyist".

As we shall see, the differences between the algebra of **V** and the Latin predecessors (which, on their part, are close to al-Khwārizmī and Abū K-);a)mil) are even greater than pointed out by Karpinski. It is therefore of some importance for our understanding of the development of European algebra to decide whether it goes back to Jacopo da Firenze (or at least some close contemporary of his) or to some late ps.-Jacopo.

The Vatican manuscript

Van Egmond [1980: 224] describes **V** in these terms:

> s. XV (c. 1450, *w*[atermark]), Holograph *libreria* treatise
>
> Paper, 4^{o} gr[ande], 286x203 mm., 72 cc num. orig. 1-59, 59-66, 69-73, plus 2 enclosing guard sheets. [...]. Single hand, a neat semi-cursive Gothic bookhand in 1 col. press-ruled 194-205x118 mm., 32 regularly-spaced lines. Dark brown ink with alternating red and blue initials, some decorated; capitals shaded in yellow on 1r, 33v-41r, lined in red on 41v-42r; some titles in red. Tables on 2v, 3r, 4v-14v outlined in red; many colored diagrams and drawings in the margins.

The incipit reads as follows:

> Incipit tractatus algorismi, huius autem artis novem sunt speties, silicet, numeratio, addictio, subtractio, <mediatio,>[3] duplatio, multiplicatio, divixio, progrexio, et radicum extractio. Conpilatus a magistro Iacobo de Florentia apud Montem Phesulanum, anno domini M^{o}CCC^{o} VII^{o} in mensis septenbris.

The rest of the manuscript is in Tuscan with a somewhat Latinizing orthography (*dicto*, *facto*, *septimo*, *scripto*, *exemplo*, etc. - none of them used quite systematically)[4]. A number of non-standard spellings also seem to reflect the Provençal linguistic environment of Montpellier, the place where the incipit tells the treatise was made, or some other northern region:

[3] Inserted in agreement with **F**, and needed in order to fill out the number of nine species. In all quotations, + , is used to indicate scribal omissions, and { } to mark superfluous passages (e.g., dittographies). Editorial comments are in [ ].

[4] We also encounter Latinizing hypercompensations: thus for instance *librectine* (fol. 4^{v}), *cictadino* (fol. 36^{r}), *soctilita* (fol 1^{r}), *tucto*, *rocto* (both regularly).

el or *lo* almost consistently instead of *il*,[5] consistently *sera* or *serra* instead of *sara* (i.e. *sarà*) and almost consistenly *mesura* instead of *misura*; mostly *de* instead of *di*, and occasionally *que* instead of *che* (both as an interrogative and as a relative pronoun); mostly *remanere* (with declinated forms) instead of *rimanere*, and mostly *vene* instead of *viene*; mostly also *doi* or *doy* instead of *due*, occasionally *dui* or *duy*; mostly *amendori*, or occasionally *amendoil-dui*, instead of *amendue*; ten, when not written with numerals, is almost always *dece*; *-ximo* is used instead of *-simo* as ordinal suffix (and *-x-* also elsewhere for *-ss-*); *lira/lire* are *libra/libre*.[6] The copyist seems to have aimed at fidelity toward his model in this and other respects: at times he corrects a spelling, even though both the new and the old spelling are present elsewhere in the text, which suggests an aspiration for orthographic accuracy (but also shows that he did commit errors on this account, some of which he will probably not have noticed); on fol. 39^{r}, where he leaves a sequence of open spaces of c. 2 cm instead of numbers, he makes a note in the margin, "così stava nel'originale spatii". The insertion of forgotten words above the line makes it plausible that the finished copy was collated with the original.[7] On fols 46^{v}-47^{r} there is evidence that not only the ultimate but also the penultimate copyist tried to be faithful: fol. 46^{v} starts by telling that a section on silver coins has been omitted by error and is inserted "de rimpecto nel sequento foglio" (it follows indeed on fol. 47^{r}) - but the organization of the page shows that this passage was not inserted after the writing of the following section on "le leghe de monete picciole", and thus that it was present (together with a mark †† indicating the location of the omitted section) in the original used by the ultimate copyist, who followed this original rather than running the risk of accumulating more errors in an attempt to repair the mistake.

Evidently, many abbreviations are used in the manuscript, including abbreviations for *libre*, *soldi*, *denari*, and *braccia*. What is remarkable in comparison with later *abbaco* manuscripts is that key terms for mathematical

[5] *Lo* (occasionally *lu*) appears in all cases where modern Italian requires it, and is used inconsistently before *s+vowel*, *d*, *m*, *p*, *q*, *r*, and *t*.

[6] **F** [ed. Simi 1995] has an occasional *el* but mostly *il*, and as far as I have noticed no *de*. I have observed some instances of *ke* but no *que* (the occurrences of *que* and *ke* in the two manuscripts are independent of each other, when parallel passages are compared). In parallel passages, *remanere* (etc.), *vene* and *mesura* in **V** correspond to *rimanere* (etc.), *viene* and *misura* in **F**, *doi/doy* in **V** to *due* in **F**, *amendore* in **V** to *ambendue* in **F**. **F** has *diece* for ten, and speaks of *lire* (and *danaio* for *denaro*; but singular *libra* or *livra*), and has none of the Latinizing spellings of **V**.

Also likely to point to a Provençal (or French or Iberian) environment is the use of *ha* (written *a*) or *si ha* instead of *c'è* in **V**. This usage is absent from **F**.

The Florence manuscript of Paolo Gherardi's algebra [ed. Van Egmond 1978], also originally written in Montpellier, replaces *il* by *el* or *lo* as does **V**, but has *rimanere* and *di*. Two and ten are written with numerals and are hence uninformative. It has some Latinizing spellings, genuine as well as hypercompensations, which may thus be taken to characterize the environment rather than individual propensities of the writers.

[7] It will come as no surprise that the elaborate initials were also inserted after the completion of the text ? but it also follows from the omission of an initial on fol. 17^{r} and of another on fol. 42^{r}. The insertion of a missing passage in the margin of fol. 48^{r} may be due to the same hand as the initials.

operations are *never* abbreviated - neither *più* or *meno*, nor *radice*, *cosa*, *censo* or *chubo*.[8] The absence even of as simple an abbreviation as *cẽso* for *censo* can only be a consequence of a deliberate choice.

Remarkable is also a less standardized technical vocabulary than in other treatises -*el diametro*, for instance, may also be both *el dericto de mezzo* and *el mino lungho*. That a number m falls short of another number n by an amount p may not only be expressed "da 7 infino in 3 menoma 4" (**21**.8, fol. 50^r) - the standard expression of **F** - but also (in parallel passages) "da 7 infino in 4 mancha 3" (*ibid.*) or "da $9^1/2$ infino in $8^1/2$ à uno" (**21**.6, fol. 49^r), with the variant "da 4 a 7 si à 3" (**21**.4, fol. 48^r).

The Latin incipit, the Latinizing spellings and a reference to Boethius's *Arithmetic* in the introduction should probably not be taken as indications that Jacopo was a university scholar, only that he moved in an environment where scholars and mathematical practitioners were in contact (other indications suggest that this was indeed the situation in Montpellier - cf. [Hahn 1982: xxi*ff*); later on he demonstrates repeatedly not to know the difference between *proportio* and *propositio*. The religious invocations of the introduction correspond well to the style of other *abbaco* writers (and of Arabic writings) but only enter the style of more scholarly work in the form of gentle parody.[9]

The structure of **V** is as follows:

1^r-1^v	**1**.Incipit and general introduction.
2^r	**2**.Introduction of the numerals and the role of *zero*.
2^v-3^r	**3**.Tabulated writing of the numbers 1-49, 50, 60, ..., 100, 200, ..., 1000, 6000[10], ..., 10000, 20000, ..., 100000, 200000, ...1000000, with corresponding semi-Roman[11] writings.
3^r-4^v	**4**.Explanation and exemplification of the place-value principle.
4^v	**5**.Introduction to the multiplication tables.
4^v-9^r	**6**.Multiplication tables, including multiples of *soldi* expressed in *libre* and *soldi*.

8 In contrast, in the tabulation of degrees of fineness of gold coins on fol. 46^r, ?meno? is abbreviated **Errore. Solo documento principale.**: ?charati 24 **Errore. Solo documento principale.** $^1/5$ per oncia?, etc. This abbreviation appears to have belonged specifically to the domain in question, cf. [Vogel 1977: 11].

9 I know of two examples: *Liber Jordani de triangulis*, a student *reportatio* of a series of lectures probably held by Jordanus himself [Høyrup 1988: 347?351], and Chuquet's trinitarian argument for the title of his *Triparty* [ed. Marre 1880: 593].

10 The row containing the numbers 2000-5000 is evidently omitted by error.

11 E.g., 600 is v̊ .

9^r-12^v	**7**.	Tables of squares; 1x1, ..., 100x100, 110x110, ..., 990x990, 1000x1000, and $1^1/2$x$1^1/2$, ..., $19^1/2$x$19^1/2$.
12^v-14^r	**8**.	Examples of divisions (9 successive divisions of 16-digit numbers by 2, then by 3, 4, ..., 19, 23, 29, 31, 37, 41, 43, 47, 48 (according to the heading the numbers that are *più necessarii*).
14^v	**9**.	Graphic schemes that serve the multiplication, addition, comparison,[12] division and subtraction of fractions (in this order).
15^r-17^r	**10**.	Examples explaining the addition, subtraction, multiplication and comparison of fractions.
17^r-18^v	**11**.	The rule of three, with examples.
18^v-19^v	**12**.	Computations of non-compound interest.
19^v-21^v	**13**.	Rule-of-three problems involving metrological conversions.
21^v-30^r	**14**.	Mixed problems, including partnership and genuine "recreational" problems.
30^v-36^r	**15**.	Practical geometry: Rules and problems involving the diameter, perimeter and area of a circle; the area of a rectangle (and, erroneously, of the regular pentagon); and the "rule of Pythagoras". With approximate computation of square roots.
36^v-42^r	**16**.	Rules and examples for algebra until the second degree.
42^r-43^r	**17**.	Rules without examples for reducible third- and fourth-degree equations.
43^r-43^v	**18**.	A grain problem of alligation type.
43^v-45^v	**19**.	Second- and third-degree problems on continued proportions (dressed as wage problems) solved without the use of *cosa-censo* algebra.
45^v-47^r	**20**.	Tabulated degrees of fineness of coins.
47^v-50^v	**21**.	Alligation problems.
50^v-$58bis^v$	**22**.	Further mixed problems, including practical geometry. In part cross-referenced variations or transformations of problems from chapters **14-15**, in part new types.[13]

12 Seeing which of two is greater, and finding the remainder; kept apart from subtraction though evidently solved in the same way.

13 In her edition, Annalisa Simi divides **F** as follows:

Incipit, Prologo	corresponding to	**V.1-2**;	Chapter III,	corresponding to	**V.10**.
Tabella 1-2,	corresponding to	**V.3-4**;	Chapter IV,	corresponding to	**V.11-13**.

In the following, passages in **V** will be referred to as **V.A**.*n* (or, if there is no doubt that **V** is spoken about, simply **A**.*n*), where **A** is the chapter number in Arabic numerals and *n* the section number within the chapter (counted in agreement with the initials). Similarly, references to **F** will have the form **F**.R.*s* (or simply R.*s*), where R is the chapter number in Roman numerals and *s* the section number within the chapter (both counted as in [Simi 1995]). Neither manuscript contains any of these numbers.

The text is characterized by interspersed personal and colloquial-pedagogical remarks - for instance:

(a) Abiamo dicto de rotti abastanza, però che dele simili ragioni de rotti tucte se fanno a uno modo e per una regola. E però non diremo più al punte. (**11**.1, fol. 17^r).

(b) Et se non te paresse tanto chiara questa ragione, si te dico che ogni volta che te fosse data simile ragione, sappi primamente ... . (**14**.19, fol. 26^r).

(c) Una torre ... sicomo tu vedi designata de rinpetto. (**15**.9, fol. 31^v; similarly *passim*).

(d) Ora non si vole agiungere inseme come tu facesti <in> quella da sopra. (**15**.10, fol. 32^r).

(e) Fa così, io t'ò anche dicto de sopra, ogni tondo, a volere sapere quanto è el suo diametro, si vole partire per 3 e $^1/_7$. (**15**.11, fol. 32^r).

(f) Et abi a mente questa regola. In bona verità vorrebbe una grande despositione; ma non mi distendo troppo però che me pare stendere et scrivere in vile cosa; ma questo baste qui et in più dire sopra ciò non mi vo' stendere. (**16**.14, fol. 40^v).

(g) Fa così, et questa se fa propriamente come quella che tu ài nanzi a quella ragione de sopra a questa, et in questa forma. Et però non me stendarò in si longho dire como feci in quella. (**21**.8, fol. 49^r).

(h) Et però ho facta questa al lato a quella,[14] che tu intende bene l'una et l'altra, et che l'una et l'altra è bona reghola. Et stanno bene. Et così se fanno le simili ragioni. (**22**.6, fol. 52^r).

A mathematical particularity of the manuscript is the use of the partnership as a functionally abstract representation of proportional sharing. In **14**.9-10 (fols 23^v-24^r) it is used to determine the shares of a heritage; in the algebra section, a sub-problem of the partnership problem **16**.3 (fol. 37^r) is represented by means of *a different* partnership; **21**.4 (fol. 48^r) introduces it as a basic tool for alligation computations, which is referred to in **21**.6 (fol. 49^r) and used again explicitly in **21**.8 (fol. 50^v); in **22**.1 (fols 50^v-51^r), a partnership problem where the participants in the *compagnia* do not enter at the same time, imagines a different partnership where the shares are the respective interests which the investments would have earned (and thus proportional to the product of investment and time). The latter type (a modest generalization) is not uncommon - I have noticed it in the fifteenth-century Provençal treatise from Pamiers [ed. Sesiano 1984: 47], with Pier Maria Calandri [ed. Arrighi 1974: 36*f*], and in a fourteenth-century *abbaco* from Lucca [ed. Arrighi 1973: 75, 77] (in the latter also with recourse to the virtual interests, whereas the others just multiply time and investment); and with Paolo Gherardi [ed. Arrighi 1987: 38]; but the general use of the structure as an abstract model I have observed nowhere else.

V also exhibits a rhetorical particularity. When explaining the reason for a particular step, it regularly ascribes to the "tu" of the text such knowledge or conditions that were originally stated by its "io" ("perché tu di' che ...", etc). Examples of this is found in all chapters which offer the occasion, that is, **14-19**, **21** and **22**.

The manuscript is beautifully illustrated. Some of the illustrations are neutral and to the point (circles with indications of the measures of diameter and perimeter, etc.), but many are not (most aberrant is a beautiful plant with flowers, stalk and root along with the rule for approximating square roots).

The Florence versus the Vatican manuscript

I have had no opportunity to examine **M**, but I have been able to confront **V** with Annalisa Simi's edition of **F** [1995].[15]

The most obvious difference is the absence of sections **16-19** and **22**; they are also lacking in **M**, cf. [Van Egmond 1980: 166*f*]. Two obvious possible explanations are at hand. Either a copyist has inserted extra material into **V** (or a precursor in the stemma), as supposed by Van Egmond; or another copyist has deleted some sections when producing a common precursor for **F** and **M** (since exactly the same sections are lacking in both of these, independent abbreviation is unlikely).

But there are differences at numerous other levels which, together with partial and complete agreements, allow us to decide the question and to conclude that **V** is fairly faithful to the original, and **F** a free adaptation.

One clue is the way the illustrations are referred to. **V** habitually refers to the illustrations and diagrams in the margin in words similar to those of quotation (c). **F** has most of the same illustrations (somewhat fewer, and none which are not in **V**), but does not always give a reference in the text; when it does, the reference is located after the solution of the problem, whereas **V** gives it after the statement. This excludes simple expansion or abbreviation and again leaves us with two possibilities: either *V* is a rewriting aiming at greater completeness and consistency, or **F** is a rewriting aiming at conciseness; so far, both possibilities remain open.

Chapter I,	corresponding to	**V.6-7**;	Chapter V,	corresponding to	**V.14**.
Chapter I,	corresponding to	**V.6-7**;	Chapter VI,	corresponding to	**V.15**.
Chapter II,	corresponding to	**V.8-9**;	Chapter VII,	corresponding to	**V.20-21**.

F contains no counterparts of **V.5**, **V.16-19** and **V.22**.

[14] In the preceding problem, the area of the circle is found as $1-{}^1/_7-{}^1/_2.{}^1/_7$ times the square on the diameter; in the actual problem, it is found as $^1/_4$ of the product of diameter and circumference.

[15] When quoting this and other published editions, I follow the orthography of the edition.

But some details imply that the manner and the illustrations of **V** are those of the original. Firstly, we may notice the following passage in **F**.VII.7 (p. 37): "simigliantemente il mostriamo materialmente per figure come se fae lo detto allegamento". In the corresponding location in **V.21**.7 (fol. 49^{v}), we read "Et simile<mente> porremo la figura. Nel modo si fa como abbiamo facto de sopra nell'altra ragione". In **V**, the promised diagram is present - but it is absent from **F**. Secondly, both **V.15**.21 (fol. 34^{v}) and the counterpart **F**.VI.17 (p. 30) contain a drawing of a tent, and both give a reference; but whereas the illustration of **V** has the conic form presupposed by the problem, that of **F** is so glaringly discordant with the description in the text that Annalisa Simi inserts a "(sic!)"

A number of other features also lead to the conclusion that **V** is close to an original of which **F** is an adaptation.

Most striking is the shift from "io" to "tu" when reasons for a step are given. As told, it characterizes all chapters of **V** which allow it (**14-19**, **21**, **22**). Of these, **16-19** and **22** have no counterpart in **F**. In the counterparts of **14** and **15** (**F**.V-VI), no similar shift is found - but in the counterpart of **21** (**F**.VII), it turns up regularly in places where it is also found in **V**, but not in those where **V** does not make use of it (except an occurrence of VII.2, the counterpart of which has been omitted from **V**).[16]

At times the two manuscripts differ in the way a problem is dressed. On at least two occasions, however, **F** betrays descent from the formulation of **V**:

(i) **V.14**.30 (fol. 29^{r}) starts "Egli è uno muro, el quale è lungho 12 braccia e alto sette. Et grosso uno et $^{1}/_{4}$". The counterpart **F**.V.34 (p. 25) has instead "egli è un terreno lo qual è ampio 12 braccia, cioè uno muro, et è alto braccia 7 ed è grosso braccia 1 et $^{1}/_{4}$". Obviously, the writer starts by changing the original wall into a terrain without having read the whole problem, and then discovers that the presence of a height makes the change impossible, and corrects himself.

(ii) **V.14**.3 (fol. 22^{r}) starts "Uno à a'ffare uno pagamento in Bolognia". In **F**.V.3 (p. 17) we find instead "I'ò a'ffare uno pagamento im Bologna" - but afterwards **F** follows **V** in the choice of grammatical person.[17]

The treatment of approximate square roots ($\sqrt{a} \approx n+{}^{d}/_{2n}$, $a = n^{2}+d$) is informative on several levels. Least decisive but still in accordance with the way illustrations are referred to is the contrast between the rather systematic habit of **V** of pointing out that this is *not* precise ("non è appunto") and the much more scattered observations of the same kind in **F**. Moreover, after having presented the rule, **V.15**.13 tells that its outcome "serà la più pressa

[16] Similar shifts seem to be very rare elsewhere. Indeed, I have only noticed one instance, namely in the late fourteenth or early fifteenth *Libro di conti e mercatanzie* [ed. Gregori & Grugnetti 1998: 50]. In most cases, however, this treatise (whose pedagogical pretensions are comparable to those of **V**) refers such explanatory information to "noi" and not to "tu" (a possibility which is also used regularly in **V**).

[17] The fact that this choice is not very consistent may have called forth the unsuccessful emendation.

radice", while **F**.VI.10 (p. 29) believes it to be "radice vera o piue pressa", as if the occasional reference to its only approximate character is mere lip service not supported by full understanding.

But there is more to square roots. **V** not only mentions repeatedly that $n+{}^{d}/_{2n}$ is merely "the closest approximation" but also shows time and again (in actual numbers) that $(n+{}^{d}/_{2n})^{2}$ exceeds a by $d^{2}/(4n^{2})$. In **15**.22 (fol. 35^{r}) it therefore gives the "improved" value $\sqrt{108} = 10^{2}/_{5} - {}^{4}/_{25}$, in **15**.25 (fol. 36^{r}) correspondingly $\sqrt{569} = 23^{20}/_{23} - {}^{400}/_{529}$ - and in **15**.20 (fol. 34^{v}) $\sqrt{33^{1}/_{3}} = 5^{7}/_{9} - {}^{4}/_{18}$.[18]

F contains no counterpart of **V.15**.22, and the counterpart of **V.15**.25 gives $\sqrt{569}$ as $23^{20}/_{23}$ without pointing out that the value is approximate (and in the following section it gives another example of the rule, still as if it were exact). Once more, this might mean that the fallacy of **V** has been added onto a sounder stem, or that it has been eliminated in **F**. **F**.VI.18 (p. 31), however, the counterpart of **V.15**.20, tells $\sqrt{33^{1}/_{3}}$ to be "5 et $^{5}/_{6}$ meno $^{17}/_{54}$ non appunto". $5^{5}/_{6}$ is obviously found in the usual way, as $5+(33^{1}/_{3} - 25)/(2.5)$. If the fallacious procedure of **V** had been used to find the correction, it would have been $(5^{2})/(6^{2})$. The actual correction instead is $(2.5^{5}/_{6})/(6^{2})$, which appears to mix the formula of **V** with the meaningful correction $({}^{5}/_{6})^{2})(2.5^{5}/_{6})$.[19] The only explanation seems to be that the writer of **F** (or a precursor) will have seen that something is wrong in the correction of **V**, and that he has tried at one point to repair by having recourse to another formula - but unfortunately remembering or applying it wrongly.

These observations should already suffice to show that **F** is a remake, and **V** much closer to what Jacopo had written in Montpellier in 1307 (as also suggested by the orthography of the two manuscripts, with all the provisos which are needed when medieval orthography is used as an argument); for stylistic convenience I shall henceforth take this result for granted. But they do not exhaust the list of characteristic differences between the two manuscripts that point in the same direction.

The use of the partnership as a functionally abstract representation of proportional sharing was mentioned above. The same idea turns up in **F**.V.10, the counterpart of **V.14**.9-10, but nowhere else in **F** - in particular not in the alligation chapter, where **V** uses it consistently.

[18] Obviously determined in this way: $\sqrt{33^{1}/_{3}}$ is found as $\sqrt{300}/_{3}$, where $\sqrt{300}$ is approximated from n = 18, which gives $5^{7}/_{9}$ as the first approximation. Since $(5^{7}/_{9})^{2} = 33^{1}/_{3}+{}^{4}/_{81}$, the fallacious rule of **15**.22 and **15**.25 would hence give the value $5^{7}/_{9}-{}^{4}/_{81}$, which is miswritten $5^{7}/_{9}-{}^{4}/_{18}$ (a similar inversion of digits is found in **14**.3, fol. 22^{r}).

[19] This formula is described by al-Qalasādī [ed., trans. Souissi 1988: 61] - and, in ambiguous terms, by ibn al-Bannā **Errore. Solo documento principale.** [ed., trans. Souissi 1969: 79].

In **V**, areas (and volumes!) are mostly expressed explicitly in *braccia quadre*, not simply in *braccia*. In **F**, this usage is less frequent - and in **F**.VI.16 the writer can be seen to be so little attached to it that he misconstructs the phrase of **V.15**.19 (fol. 34ʳ), "vo' sapere quante braccia quadre è tucto questo terreno", as "dimmi quanto è tutto quel terreno quadro", even though the terrain in question is *not* a square but a regular pentagon.[20]

V, as pointed out, is stuffed with personal and colloquial-pedagogical remarks. **F** has much fewer of these, and their style is less colloquial. Of the quotations (a) to (h), only (c) has a genuine counterpart (**F**.VI.8, p. 28): "io ti mostro qui appresso la forma per meglio intendere". (e) is reduced to "Fa cosie. Et quest'è la sua propria regola. Parti 100 per 3 1/7 in questo modo" (**F**.VI.10, p. 28). (f) and (h), of course, but only these, belong in chapters without any counterpart in **F**. Similarly, the general introduction to the multiplication tables (**V.5**), the specific introduction to the "librettine magiori" (**V.6**.2) and the introduction to the divisions (**V.8**.1) are absent from **F**. Absent from **F** are finally a number of metatheoretical explanations - for instance the explanation which **V.15**.2 gives after telling how to find a circular diameter from the perimeter, and vice versa: "Et se volissi sapere per che cagione parti et moltiplichi per 3 e $^1/_7$, si te dico che la ragione è perché ogni tundo de qualunqua mesura se sia è intorno {intorno} 3 volte et $^1/_7$ quanto è el suo diametro, cioè el diricto de mezzo. Et per questa cagione ày a moltiplichare et partire como io t'ò dicto de sopra".

In general, in cases where mere copying mistakes and failing understanding can be excluded, the replacement of an error by a good solution is more likely to occur than the inverse process when a computational text is corrected. Comparison of these two problems might therefore seem to speak against the primacy of **V**:

V.11.4 Ancora diremo chosì. 7 libre di tornesi vagliono 9 libre de parigini. Che varrano 120 libre de tornesi ? Fa così como de sopre abiamo dicto: 120 libre via 9 libre de parigini fanno 1080 libre de parigini; et parti per 7 libre de tornesi, cioè, parti 1080 in 7, che ne viene 154 libre et 5 soldi et 8 denari e $^4/_7$. Et cotanto diremo che vagliono le 120 libre de tornesi, cioè libre 154, soldi 5, denari 8$^4/_7$ de parigini.

20 Both manuscripts find the area as $3s^2 - s$, s being the side. This nonsensical formula is certainly derived from the formula for the n'th pentagonal number, $^1/_2.(3n^2-n)$ by omission of the halving.

I have noticed Jacopo's formula in Tommaso della Gazzaia's *Pratica di geometria e tutte misure di terre* [ed. Nanni 1982: 24*f*] from c. 1400 (where, however, somebody has added the note "non e vera"). Tommaso's first example is based on the side 8, as both **V** and **F**. Other problems in Tommaso's treatise also repeat Jacopo with the same numerical parameters. The "correct" formula $^1/_2.(3s^2-s)$ is found in the *Geometria incerti auctoris* [ed. Bubnov 1899: 346], in the agrimensor treatise of Epaphroditus and Vitruvius Rufus [ed. Bubnov 1899: 534], and in *Artis cuiuslibet consummatio* [ed. Victor 1979: 158].

F.IV.3 Ancora diremo: 7 tornesi vagliono 9 parigini, dimmi quanto varranno le 120 di tornesi. Fa cosie. Die: poichè 7 tornesi vagliono 9 parigini, dunque 7 soldi di tornesi vagliono 9 soldi di parigini et 7 lire di tornesi vagliono lire 9 di parigini. Dunque multiplicha 9 via 120 lire de parigini, fanno lire 1080 e parti lire 1080 per 7, che nne viene lire 154 e soldi 8 e denari 6 e $^6/_7$. Et diremo che 100 lire di tornesi varranno lire 154 e soldi 8 et denari 6 et $^6/_7$, d'uno danaio et è fatta apunto.

Of the two incompatible answers, **V** provides the one that is "apunto", and the rule of thumb formulated above would therefore speak against the primacy of **V**.

However, rules of thumb *are* rules of thumb, and presuppose a particular context that they do not make explicit. Here, the presupposition - that one of the texts corrects the calculation of the other - is invalid. The originator of **F** does not *correct the calculation* of **V** but *the approach*, making specific translations at each of the levels *libre* and *soldi*. He therefore makes the calculation anew - and errs.[21] Probably because he presupposes his own method to be better, the *Verballhornung* seems not to worry him.

Evidently, all these arguments (and others of the same kind, which it would take up too much space to list) only show that **F** is the outcome of a rewriting of an original to which **V** is much more faithful[22] *in the chapters which are present in both versions*. They do *not* exclude that the chapters on algebra (**V**.16-19) and the final sequence of mixed problems (**V**.22) be additions.

However, other arguments speak strongly against this possibility. At first we may look at the contents of chapter **22**, which from the above general description seems to overlap chapters **14-15**. At closer inspection, however, the apparent overlap turns out to consist of duly cross-referenced variations and supplements; no single repetition can be found. This would hardly have been the case if a later hand had glued another problem collection onto an originally shorter treatise, given the frequency of whole-sale borrowings of problems between different *abbaco* writings.

21 The error is $^1/_7$ of a *libra*, and thus goes back to an error of 1 *libra* before the division by 7 - 2 *libre* have been transferred to the level of *soldi* as 60 instead of 40.

22 If **V** were also the outcome of a process of rewriting, if would certainly contain passages in which inconsistencies betrayed dependency on something close to **F**. I have noticed none (apart from the corrupt **V.14**.32, which appears to mix up two numerically different versions of the same problem).

This does not mean that no inconsistencies can be found in **V** - but the only ones I have observed are shared with **F**, and point to the Arabic world. Firstly, a distinction is mentioned in the introduction (fol. 1ᵛ) between "rocti sani e rocti in rocti", where "rocti in rocti" can hardly refer to anything but the *partes-de-partibus* usage of Arabic mathematics; but such "parts of parts" are never used afterwards. Similarly, the claim of chapter **8** that the prime numbers are the "most necessary" divisors is not born out by the calculations in the rest of the treatise; but it corresponds to a method that is prescribed by al-Qalasai [ed. Souissi 1988: 42] when he discusses proportional sharing - namely to add all the shares and to resolve the sum (which is going to serve as divisor) into factors; cf. also the contribution of Ezzaïm Laabid to the present volume.

To this comes the homogeneity of **V**. On all levels - orthography, vocabulary, discourse, pedagogical style - the treatise is a seamless whole, also on points where it differs from **F** or other *abbaco* writings (e.g., the *io/tu* shift and the way diagrams are referred to). The same holds for the computational techniques when there is a choice, and for the mathematical approach (for instance the use of the partnership model, and the ever-recurrent emphasis that the approximate values for square roots *are* approximate). All this in itself does not necessarily mean that everything was written originally by the same author; but if it was not, it would have required a strong harmonization and reformulation by a later hand, as has happened to the text as we find it in **F**. But a harmonization of this kind would also have affected the chapters that have a counterpart in **F**, and at some points we would certainly have found incongruities that betrayed the departure of **V** from an original stem which in these passages was closer to **F**. We must therefore conclude that whatever harmonization occurred to the text we find in **V** occurred *before* the text developing into **F** split off from the stem.

This still does not prove definitively that the chapters **16-19** and **22** were part of the treatise written by Jacopo in 1307. But their absence from **F** and **M** provides no evidence that they were not. As we shall see, at least the algebra must be dated well before Paolo Gherardi's work from 1328. All in all, the most reasonable assumption is thus that it was part of the original treatise.

Related algebraic writings

As we remember, one of the reasons that caused Van Egmond to assume a late date for the algebra of **V** was the presence of both third- and fourth-degree equations, where Paolo Gherardi's treatise from 1328 (henceforth **G**) has only third-degree equations. **V**, on the other hand, has only reducible cases, whereas Gherardi has irreducible cases as well; **V**, moreover, gives the rules without examples, while **G** illustrates all rules by examples. All in all, comparison combined with a tacit premise of cumulativity certainly does not speak unambiguously in favour of the primacy of **G**. But the very intricacy of the question suggests that closer investigation of the higher-degree problems may provide crucial information.

Two other texts can also profitably be compared with the sequence of higher-degree equations in **V** and **G**. One is an *abbaco* manuscript from Lucca from c. 1330 ([ed. Arrighi 1973], already referred to above), a conglomerate written by several hands. Its fols 80^v-81^v (pp. 194-197) contains a section on "le reghole dell'aligibra amichabile", which will be designated **L**; another section on "le reghole della chosa con asenpri" is found on fols 50^r-52^r (pp. 108-114; henceforth **C**). The other text is a *Trattato dell'Alcibra amuchabile* [ed. Simi 1994] known from a manuscript from c. 1365. The *Trattato* is part of Ricc. 2263 and is itself a composite

consisting of three clearly separate treatises.[23] Of interest in the present connection is the second of these (fols 28^r-34^r), which I shall designate **A**.

Below follows a list of the "cases" present in these five works or chapters, with indication of the order. If a rule is present for the case, it is marked R if true, X if false; the presence of examples is indicated E, marked by subscript digits (E_{12} thus indicates that two examples are given; E_1 and E_2 in the same row but different columns indicate that examples differ, E_1 and E_1* that they are identical apart from a numerical variation). The letters "p" and "n" indicate whether the division by which the equation is normalized is expressed as "partire per" or "partire in"; we shall see that this "neutral mutation" is an interesting parameter. *K* stand for *cubo*, *C* for *censo*, *CC* for *censo di censo*, *t* for *cosa*, *n* for *numero* (in whatever spellings the manuscripts use), and Greek letters for coefficients (implied by the reference to the plurals *cubi*, *censi*, and *cose*).

23 The first of these (fols 24^r-26^r) carries the heading "Incomincia il primo trattato dell'alcibra amuchabile" and gives rules for the multiplication of signed entities and binomials. Fols 26^v-27^v treat of non-mathematical subjects. The second treatise, fols 28^r-34^r, starts without any heading, "Quando le cose sono iguali al numero". Fol. 34^v is left blank, and the third treatise - fols 35^r-50^v, a collection of problems - starts again without any heading, "Fammi di 10 due parti ...". In the first and third treatise we see the incipient transformation of syncopated into symbolic algebra; according to the edition, the second contains no traces even of syncopation.

Case	V	G	L	C	A
$\alpha t = n$	1.R,E_{12},n	1.R,E_1*,n	1.R,E_1,n	1.R,E_1*,**p**	1.R,E_{12},n
$\alpha C = n$	2.R,E_1,**p**	2.R,E_2,n	2.R,E_2,n	2.R,E_2*,n	2.R,E_1,**p**
$\alpha C = \beta t$	3.R,E_1,**p**	3.R,E_1*,n	3.R,E_1*,**p**	3.R,E_2,**p**	3.R,E_1,**p**
$\alpha C + \beta t = n$	4.R,E_{12},n	4.R,E_1*,n	4.R,E_1*,n	4.R,E_1**, n	4.R,E_{12},n
$\beta t = \alpha C + n$	5.R,E_{123}, n	5.R,E_2*,n	5.R,E_2**, **p**	5.R,E_2*** ¶,n	5.R,E_{123}, n
$\alpha C = \beta t + n$	6.R,E_1,n	6.R,E_2,n	6.†	6.R,E_3,n	6.R,E_1,n
$\alpha K = n$	7.R,**p**	7.R,E_1,**p**	7.R,n	7.R,**p**	7.R,E_1,**p**
$\alpha K = \beta t$	8.R,**p**	9.R,E_1,**p**	8.R,n	8.R,**p**	8.R,E_1,**p**
$\alpha K = \beta C$	9.R,**p**	10.R,E_1,**p**	9.R,**p**	9.R,**p**	9.R,E_1,**p**
$\alpha K + \beta C = \gamma t$	10.R,n	15.R,E_1,n	10.R**,**p**	14.R,n	15.R,n
$\beta C = \alpha K + \gamma t$	11.R,n		11.R,n	15.R,n	16.R,n
$\alpha K + \gamma t = \beta C$					14.R,E_1,n
$\alpha K = \beta C + \gamma t$	12.R,n	11.R,E_1,n	12.R††,n	16.R,**p**	10.R,E_1,n
$\alpha K = \sqrt{v}$		8.R,E_1,**p**			11.R,E_1,**p**
$\alpha K = \beta t + n$		12.X,E_1,n			12.X,E_1*, n
$\alpha K = \beta C + n$		13.X,E_1,n			13.X,E_1,n
$\alpha K = \gamma t + \beta C + n$		14.X,E_1,n			
$\alpha CC = n$	13.R,n		13.R,**p**	11.R,**p**	17.R,n
$\alpha CC = \beta t$	14.R,**p**			12.R,**p**	18.R,**p**
$\alpha CC = \beta C$	15.R,**p**			13.R,**p**	19.R,**p**
$\alpha CC = K$	16.R,**p**			10.R,**p**	20.R,**p**
$\alpha CC + \beta K = \gamma C$	17.R,n				21.R,n
$\beta K = \alpha CC + \gamma C$	18.R,n				22.R,n
$\alpha CC = \beta K + \gamma C$	19.R,n				23.R,n
$\alpha CC = n$	20.R,n				24.R,n
$\alpha CC = \beta t$	7.R,n				
$\alpha CC = \beta C$	8.R,**p**				
$\alpha CC = \beta K$	9.R,**p**				
$\alpha K = \beta t$	10.R,**p**				
$\alpha CC + \beta K = \gamma C$	11.R,n				
$\beta K = \alpha CC + \gamma C$	12.R,n				
$\alpha CC = \beta K + \gamma C$	13.R,n				
$\alpha CC + \beta C = n$	14.R,n				

* The statement has a lacuna, and should read "Trouami 2 numeri che tale parte sia l'uno dell'altro come 2 di 3 e, multiprichato il primo per se medesimo et poi <per> quello numero faccia tanto quanto e più 12".

† Absent; but since the ensuing text refers to "6 reghole", this is clearly by involuntary omission.

** The rule should read "Quando li chubi <e li censi> sono egualj alle cose [...]".

†† The rule should read "Quando li chubi sono egualj <a' censi> e alle chose [...]".

¶ In the short collection of further illustrative examples, **C** also has the problem E_1 of **V**.

The first obvious observation is that the distribution of divisions *per* versus divisions *in* is strikingly similar in all cases. To some extent this *may* be determined by the subject-matter - for some obscure reason (if not by pure accident), normalizations *per* mostly occur when only two powers are involved. But this does not explain all, and the agreement still shows that the five texts are closely related.

Most kindred are **V** and **A**, where the agreement on this account in complete in all shared cases. There is also full agreement in the examples that illustrate these (and it should be observed that the examples are not formulated in terms of *censi* and *cose*, and only to a limited extent in purely numerical terms (as **V.16**.7, "Trovami 2 numeri che siano in propositione sì como è 4 de 9. Et multiprichato l'uno contra l'altro faccia quanto ragionti inseme", and the like). There is no doubt that one depends closely on the other, or on a very near parent - the only question concerns priority.

The passage that most clearly shows that **A** is derived from a predecessor of **V** is found in the example E_2 for the case $aC+bt = n$ (**V.16**.10, fol. 39^r; **A**, fols 29^v-30^r, p. 25*f*). In **V** we find

> Et però di': multiprichare radice de 54 meno 2 via radice de 54 meno 2. Et cotanto varrà el censo. Che in verità, radice de 54 meno 2 via radice de 54 meno 2, fa 58 meno radice de [24] et abbiamo che vale el censo 58 meno radice . Et noi ponemo avesse el primo uno censo. Dunqua vene ad avere 58 meno radice de . <Ora sappi el secondo, che ponesti ch'avesse $^1/_4$ censo e $17^1/_2$ numeri. Adunqua piglia el $^1/_4$ de 58 meno radice de 864> ch'è $14^1/_2$ meno radice de 54, sopra el quale vi giongi $17^1/_2$; fanno 32 mino la radice de 54. Et così abiamo che el primo à 58 meno la radice de . Et el secondo homo à 32 meno radice de 54.

The corresponding passage in **A** runs as follows:

[24] Instead of «864», the ms leaves open c. 2 cm. In the margin the copyist writes the commentary «così stava nel'originale spatii».

> Abiamo che vale la cosa radicie di 54 meno e 2; vie radicie di 54 meno 2, e cotanto varrà il cienso, che 'n verità fa 58 meno radicie di 864. E nnoi ponemo che 'l primo avesse uno cienso. Dunque avrà il primo 58 meno radicie di 864. Ora sappi il secondo, che ponesti ch'avesse $^1/4$ cienso e $17^1/2$ numeri. Adunque piglia il $^1/4$ di 58 meno radicie di 864, ch'è 14 e $^1/2$ meno radicie di 54, sopra il quale giungni $17^1/2$, fanno 32 meno la radicie di 54. E così abiamo che 'l primo à 58 meno radicie di 864 e'l secondo huomo à 32 meno radicie di 54.

The passage in < > in **V** is filled out according to the words of **A**, but in the orthography of **V**. That it fits perfectly into the rest only confirms that the two texts are close to each other. The empty spaces in **V** are more informative. They demonstrate that **V** descends via attempted faithful (though, as we see, not always actually faithful) copying from a prototype (henceforth **V'**) prepared in the original process of computation (at least in as far as this problem is concerned) - a manuscript where the author left open the spaces where he might insert the result when he had calculated 16x54, but then forgot to do so. **A**, instead, either derives from **V'** via an intermediate manuscript **A'** where the calculations were performed - or they are made directly in **A**. In any case, **V** is not derived from neither **A** nor the hypothetical **A'**.[25]

A is also a witness of other more fundamental innovations. Firstly, it introduces examples for several of the higher-degree cases, all of which appear in **V** "senza niuna dispositione", "without any exposition", that is, without illustrative examples. These are all facile in construction, in contrast to many of those that illustrate the second-degree cases. As an example we may look at the illustration of the case "li chubi sono iguali a' ciensi ed alle cose" (fol. 32^v, p. 30):

> Trouami 3 numeri che sieno in proporzione insieme come 3 di 4 e come 4 di 5 e, multiprichato lo primo per se medesimo e poi per lo numero, faccia tanto come lo secondo multiprichato per se medesimo e posto in suso lo terzo numero.

The first number is taken to be "3 cose" and the others "4 cose" and "5 cose", respectively, which yields the equation

25 Comparison of other parallel but diverging passages lead to the same conclusion ? for instance the remarks that close the discussion of the double solution of the case $\beta t = \alpha C+n$. In **V.16**.14 (fol. 40^V) they run

> Et abi a mente questa regola. In bona verità vorrebbe una grande despositione; ma non mi distendo troppo però che me pare stendere et scrivere in vile cosa; ma questo baste qui et in più dire sopra ciò non mi vo' stendere.

To this correspond in **A** (fol. 31^r, p. 27) the following inconsistent passage (no reader was ever asked to "keep in mind" that the author should have given a more thorough explanation):

> Ed abi a mente che questa reghola vorebe una grande dispusitione, ma non mi ci distendo tropo che melo pare scrivere multa cosa e questo basti.

A, or its original, as we see, uses (some precursor of) **V**, and squeezes two sentences into one - that **V** should be able to take a single meaningless period from some **A'** and expand it meaningfully into two as here is quite unlikely). And whereas the author of **V'** is tempted to elaborate the argument, the writer of **A** or its original finds his source too loquacious.

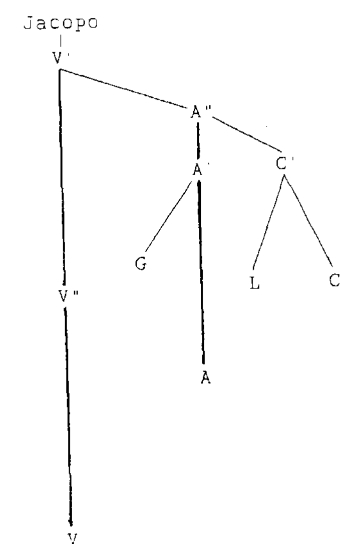
$$(3 \text{ cose})^3 = (4 \text{ cose})^2+(5 \text{ cose})\ .$$

Since no care is taken to avoid irrational solutions, it is obviously easy to construct suitable illustrations for all cases.

We also find some additional cases - not only $aK = \sqrt{n}$ but also the two irreducible cases $aK = bt+n$ and $aK = bC+n$, both solved by means of the wrong formula $t = \sqrt{\frac{n}{\alpha} + (\frac{\beta}{2\alpha})^2} + \frac{\beta}{2\alpha}$, and illustrated by means of examples that lead to unhandy irrational solutions that do not invite to verification.

The same 3 additional cases are found in **G**, with the same rules and the same examples. In **G**, we notice, *all* rules are provided with illustrations, and in all third-degree cases where **A** has an example it recurs in **G**. Again we have to ask whether **A** is derived from **G** (or some **G'** close to **G**) or vice versa.

The treatment of the case $\beta t = \alpha C+n$ solves this problem. **G** has only one of the three illustrations of this case that are found in both **V** and **A**, - in a numerical variation E2** that leads to an irrational solution. **V** and **A**, instead, share a nice integer solution - or rather two, because they have and explain the existence of a double solution, which **G** ignores though having it in the rule. Moreover, as we have seen in note 25, the explanations given in **V** and **A** are closely related and not independent discoveries of the same mathematical fact. No doubt, therefore, that Paolo Gherardi borrows from a predecessor **A'** of **A** in which the innovations with respect to **V'** were already present.



We notice that **A** has examples of the rules until a certain point, and then rules only. **G**, as observed above, has examples to all rules, but with one exception, all its cases are also in **A**, that is, already in **A'**. The exception is the case $\alpha K = \gamma t+\beta C+n$, which is solved as if it had been $\alpha K = (\gamma+n)t+\beta C$ (or, with the same wrong formula, as $\alpha K = \beta C+(\gamma+n)$). This is a fallacy so to speak of a higher order than the others, and we may assume that it was added by Paolo Gherardi himself or somewhere between **A'** and **G**.

Next we shall have to look at **L** and **C**. None of them have examples for the higher-degree cases, in which respect they are closer to **V** than **A**. **C** is the one that comes closest to **V** and **A** in the distribution of divisions *per* respectively *in*, and also the one that has the largest number of rules (but like **L** none that are not found in both **V** and **A**). However, there are some differences in the *per/in* distribution, and some of the illustrative examples for the second-degree cases are also different. Given the complete agreement between **V** and **A** on these and many other accounts, **A** cannot descend from neither **L** nor **C**; they will have diverged from the line connecting **V'** and **A'**. Since **L** mostly differs from **A** on the points where **C** differs and has none of the rules which are omitted in **C**, they are likely to derive from a common precursor **C'**.

All in all, we are led to the stemma shown above for the five expositions of algebra (**F** and **M**, containing no algebra, do not enter).[26] **V"** designates the manuscript in which the list of silver coins was first displaced, and whose explanation of this was copied in **V**; the thick line signifies (attempted) faithful copying, thin lines more or less creative use and re-elaboration.

Since Jacopo wrote in 1307 and Paolo Gherardi (**G**) in 1328, and since **C** and **L** are dated around 1330, **A'** and *a fortiori* **A"**, the point where divergence toward **C** and **L** begins, have to fall before that date; this constitutes no absolute proof that **V'** - the point where the algebra got into Jacopo's treatise and was harmonized stylistically with the rest - is the original writing of the treatise. But so little time is left, allowing a reasonable distance between the certainly different points **A"**, **A'** and **G**, that it is the only reasonable assumption, which I shall therefore adopt in

26 Further arguments for this stemma follow from comparison of the precise words of the different texts. As an illustration, we may first quote the rule for the case $\alpha K = \beta C+\gamma t$:

V: dei partire <ne> li chubi et poi dimezzare li censi, et multiplichare per se medesimo et agiungere alle cose, et la radice dela summa più el dimezzamento de' censi vale la cosa.

L: si vuole partire ne' chubij e poj dimeççare ciensi e multiprichare per se medesmo e giugnere sopra alle chose, e lla radicie della somma più il diççamento de' censi vale la cosa.

C: dovemo partire per li chubi e poj dimeççare i censi e multiprichare per se medesmo e porre sopra le chose, acciò che nne viene sarae radicie di quello e più lo dimeççato de' censi, e chotanto varrae la cosa.

A: dei partire ne' chubi, poi dimezare i ciensi e multiprichare per se medesimo e giungniere alle cose e la radicie della somma più il dimezamento de' ciensi vale la cosa.

G: dovemo partire ne chubi e poi dimezare le cose [*sic*, read "censi"] e multiplicare questo dimezamento per se medesimo e porre sopra le cose, e ℝ di quella somma che fara piue lo dimezamento varra la cosa.

A, we note, is very close to **V**. **L**, **C** and **G** diverge in different directions although with some similarity between **C** and **G**.

The case $\alpha C = \beta t+n$ (forgotten in **L**) is treated in this way:

V: se vole partire nelli censi, et poi dimezzare le cose, et multiplicare per se medesmo et giongere al numero. Et la radice dela summa più el dimezzamento dele cose vale la cosa.

C: dovemo partire ne' censi, e poj dimeççare le cose e multiprichare per se medessmo e pollo sopra il numero, e sarae radice di quello e piue lo dimeçamento delle cose; e cotanto vale la cosa.

A: dei partire ne' ciensi e poi dimezare le cose e multiprichare per se medesimo e giungniere al numero; e la radicie della somma più il dimezamento delle cose vale la cosa.

G: de partire ne censi e poi dimezare le cose e multipricare per se medesmio a ragiungnere sopra lo numero, e la radice di quello piue lo dimezamento dele cose vale la cosa.

Here, **A** shares one of the three changes from **V** to **G** (*se vole > dei*, but neither *giongere al > ragiungnere sopra* nor *dela summa > di quello*). This time, **G** and **C** diverge in totally different ways.

what follows. If I am mistaken, it is a least certain that **V'** has to be located very soon after 1307, and well before 1328.

Jacopo's "innovations"

I am publishing an edition and translation of the algebraic chapters of **V** elsewhere [Høyrup 1998a], for which reason I shall only summarize their characteristics here.

Globally, it is remarkable that Jacopo's algebra does not share a single example *and only a single rule* with the Latin algebras - that is, with the sum-total of Robert of Chester's and Gherardo da Cremona's translations of al-Khwārizmī and the revision perhaps to be ascribed to Guglielmus de Lunis; the translation of Abū Kāmil; and Leonardo Fibonacci's *Liber abbaci* and *Pratica geometrie*.[27]

It may amaze that only a single rule - namely for the case $\alpha t = n$ - is shared with the Latin works. The explanation is that the rules of the latter for the second-degree cases presuppose problems to be normalized[28] (the first-degree problem is evidently non-normalized - if it were not, the statement of the problem would already *be* the solution). Jacopo, as the other treatises discussed above, presupposes problems to be non-normalized, for which reason the first step of all rules is a normalization.

The Latin algebras (with the exception of the *Pratica geometrie*, whose algebra is subordinate and not the subject-matter proper) illustrate the rules by examples formulated in terms of the representation - "the *census* and 10 roots are made equal to 39", etc. None of Jacopo's problems are of this sort (nor are those of **A**, **C**, **L** and **G**). Some of Jacopo's (and all of the higher-degree problems of the others) are pure-number problems ("find two numbers which are in the same proportion as ...", etc.); but others are dressed as real-life problems concerned with partnerships, commercial profit from travels, etc. One of these, strikingly, deals with the square root of an amount of real money.[29] This problem type is evidently the origin of the Arabic *māl-jidr*-techniques. With al-Khwārizmī and Abū Kāmil, however, it only survives as the standard representation, and for this reason it is also absent from the Latin algebras (which translate *māl* correctly as *census* or *substantia*, but give no hint that these terms should be understood literally).

27 Ed., respectively, [Hughes 1989], [Hughes 1986], [Kaunzner 1986], [Sesiano 1993], [Boncompagni 1857a] and [Boncompagni 1862].

28 Many of their examples are certainly non-normalized, and then the texts tell how to reduced them to normalized form; but the explicit *rule* only applies when this form has already been brought about.

29 "E sonno due homini che ànno denari. Dice el primo al secondo: Se tu me dessi 14 de toi denari, che io li racchozzasse co' mey, io arei 4 cotanti de te. Dice el secondo al primo: se tu me desse la radice de toy denari, io arei denari 30" (**16**.10, fol. 39^r).

As most fourteenth-century *abbaco* algebras, Jacopo's contains no geometric proofs (nor any other argument) for the validity of its rules. Even this is of course in contrast to all the Latin algebras.[30]

Already Karpinski observed that Jacopo's order of the first six cases differs from the traditional al-Khwārizmīan order, which is also the order of Abū Kāmil's *Algebra*:

(1) $C = \beta t$, (2) $C = n$, (3) $\alpha t = n$, (4) $C+\alpha t = n$, (5) $C+n = \beta t$, (6) $\beta t+n = C$

The order of the *Liber abbaci* is different: 1-2-3-4-6-5. That of Jacopo (and of **A**, **G**, **L** and **C**) is also different: 3-2-1-4-5-6.

A final point on which Jacopo's algebra differs strikingly from the Latin translations is of course the *systematic* inclusion of reducible higher-degree cases (both Abū Kāmil and the *Liber abbaci* comprise bi-quadratic problems, but they offer no systematic treatment).

There is still a tacit tendency within the historiography of mathematics to identify Arabic algebra exclusively with what can be found in the works belonging to the "high" tradition: al-Khwārizmī, Thābit ibn Qurrah, Abū Kāmil, al-Karaji's *Fakhri*, al-Khayyāmī - and in practice with al-Khw˜);a)rizmī and Abū Kāmil alone. On this background, the presentation of the subject in **V** is so different from "Arabic algebra" and so innovative that the doubt as to its ascription to an otherwise unknown writer from 1307 becomes understandable - not least because it avoids all the erroneous rules that flourish in most *abbaco* algebras from Paolo Gherardi to Piero della Francesca.

But this identification is false, and Jacopo's apparent innovations can also be traced in the Arabic world.

Let us first look at the order of six basic cases. The order 1-2-3-4-5-6 is not only the order of al-Khwārizmī and Abū Kāmil (Arabic as well as Latin translations); it is also the order of Thābit ibn Qurrah's *Verification of the Problems of Algebra through Geometrical Demonstrations* [ed., trans. Luckey 1941: 105-107] - only 4-5-6); ibn al-Bannā's*Talkhīs* [ed., trans. Souissi 1969: 92]; ibn al-Yāsamīn's *Urjuza fi'l-jabr wa'l-muqābalah* (paraphrase in symbols in [Souissi 1983: 220-223]); and ibn Turk [ed. Sayılı 1962: 145-152] (1-4-5-6 only). It is certainly to be regarded as the classical order.

Al-Karajī, however, has the sequence 3-1-2-4-5-6, both in the *Kāfī* [ed., trans. Hochheim 1878: III, 10-13] and in the *Fakhrī* [paraphrase Woepcke 1853: 64-71]; he is followed by al-Samaw al, al-Kāsī, and by Bahā al-D˜);i)n[ed., trans. Nesselmann 1843: 41*ff*]. Finally, Jacopo's arrangement is told in al-Māridīnī's commentary to ibn al-Yāsamīn's *ᶜUrjūza* from c. 1500 [Souissi 1983: 220] to be what is used in "the East", and it is indeed the order of al-Miṣṣīṣī al-Bīrūnī, al-Khayyāmī and Saraf al-Dīn al-Ṭūsī [Djebbar 1981: 60]; but it is also followed by al-Qurasi (thirteenth-century, born in Andalusia, active in Bugia in Algeria) [Djebbar 1988: 107]. It thus appears to have been quite widespread in Jacopo's epoch.

30 I disregard the brief presentation of "gleba mutabilia" in *Liber Alchorizmi de pratica arismetice* [ed. Boncompagni 1857b: 112*f*]. The algebra section is not in Allard's partial edition of the *Liber Alchorizmi* [1992] but present in manuscripts that are as far as possible from each other in the stemma - see [Høyrup 1998c: 16 n.7] and thus doubtless part of the original work and no interpolation. But it appears to have had no impact whatsoever.

Widespread in early second-millennium Arabic algebra was also the solution of reducible higher-degree equations like Jacopo's cases 7-14;[31] however, apart from the solutions by means of solid geometry due to al-Khayyāmī and others, Arabic algebra did not treat the irreducible cases; Van Egmond is certainly right when asserting [1978: 163] that "no Arabic algebraist could have written a treatise so full of elementary errors" as **G**, and that "Leonardo Pisano was far too intelligent to have written such foolishness".

The question of normalization is intricate. As pointed out, the second-degree cases are all normalized in the Latin algebras; so are the rules which Thābit and ibn Turk quote in their proofs.[32] In the published Arabic text of al-Khwārizmī [ed. Musarrafa & Aḥmad 1939], in contrast, they are non-normalized.

At close analysis, however, this Arabic text turns out to have been submitted to at least three revisions since the stage of which Gherardo's translation is a witness (see [Høyrup 1998b]). Given Gherardo's grammatical faithfulness we may be quite confident that even al-Khwārizmī's cases were normalized; the extant text is a witness of the process in which living, practised algebra in the Arabic world drifted toward inclusion of the normalization step in the explicit rule.

Al-Karajī's treatises show us the same process at work. The *Kāfī* [trans. Hochheim 1878: III] only gives rules for the three simple cases, but all of these are for the non-normalized form.[33] According to Woepcke's paraphrase of the *Fakhrī* [1853: 64*ff*], the same principle prevails here.

The *Kāfī* exhibits other interesting features. First of all, it gives no geometric proofs; this characteristic recurs in the Maghreb tradition, and with Bahā6 al-Dīn. Secondly, its use of the verbs *jabara* and *qabila* (habitually translated "restore" and "oppose" in the context of algebra, and the very verbs which in nominalized form give the technique its name, *Al-*

31 See al-Karajī's *Fakhrī* [paraphrase Woepcke 1853: 71*f*] and ibn al-Bannā **Errore. Solo documento principale.**'s *Talkhī* [ed., trans. Souissi 1969: 96] - and in general [Djebbar 1981: 107*f* and *passim*].

32 Ed., respectively, [Luckey 1941: 110-112] and [Sayılı 1962: 144-153].

33 The composite cases are represented by examples only and without a general formulation.

jabr wa'l-muqābalah) differs from the usage established by al-Khwārizmī[34] and seems to reflect pre-al-Khwārizmīan ways of speaking[35], as indeed confirmed by Abū Bakr's *Liber mensurationum* [Høyrup 1996: 50*f*], - and if we go to Jacopo we discover that he uses *ristorare* in the same way.[36]

Irrespective of the level which he attains in the *Fakhrī*[37], this terminological peculiarity of al-Karajī's works show that his starting point is the "low" or "non-scholarly" current to which also a Bahā'al-Dīn belongs - the current which had never felt the need to provide "Greek-style" proofs for its algorithms, and which (with all the provisos that this notion may ask for) came to dominate the mathematics of the *madrasah* in the era when scientific activity was "naturalized" in Islam [Sabra 1987]. This link of the *Kāfī* to the "low" tradition is confirmed by its practical geometry - see [Høyrup 1997, *passim*].

Every characteristic of Jacopo's algebra so far discussed thus points to the "low", non-Grecisized register of Arabic algebra. The only possible trace of a tie to Gherardo's translation or to Fibonacci is indeed of terminological character: the choice of *census*, transformed into *censo*, as the equivalent of Arabic *māl*. But even this trace is not fully unambiguous. The same choice was indeed made in the non-algebraic *Liber augmenti et diminutionis* [ed. Libri 1838: I, 304-371], and it may thus have been more routinely than it seems to us.

Nonetheless, we cannot exclude that Jacopo's use of *censo* derive from the Gherardo-Fibonacci-tradition. But his apparent innovations are clearly not innovations within this tradition - not primarily because they are borrowed (as they clearly are) and not innovations but because they do not fall "within this tradition". Instead, they - and Jacopo's whole treatment of algebra - constitute the earliest evidence of the establishment of a *new* algebraic tradition, borrowing once again from the Arabic world - not, however, from the classical treatises as the Latin works had done, but from the living "low" tradition (which was certainly also exploited by the all-devouring Fibonacci, but not as his sole resource).

34 In this usage, "restoration" means addition to both sides of an equation, which cancels a subtractive member; "opposition" means subtraction from both sides.

35 Cf. [Saliba 1972]. According to the *Kāfī* [trans. Hochheim 1878: III, 10], *al-jabr* is the elimination both of factors and of subtracted members (added members are not mentioned explicitly), which leads to *al-muqābalah*, "opposition" as two sides of a reduced equation. The *Fakhrī* has the same usage, but includes explicitly the elimination of added terms under *al-jabr* - cf. quotation and commentary in [Woepcke 1853: 64].

36 One example of the non-al-Khwārizmīan use (among several) is **V.16.13** (fol. 40ʳ), "Ristora ciascheuna parte, cioè de cavare 24 cose de ciascheuna parte". I suspect that Jacopo's "equation" (*raoguaglamento*, **16**.13, fol. 40ᵛ) may correspond to *al-muqābalah*.

37 "Low" and "high", indeed, do not refer to any measure of quality but to social prestige - and, in the actual case, to the social prestige which *our* world ascribes to various types of scientific activity.

But we may say more. If we compare Jacopo's explanatory style (as exemplified by the remarks (a)-(h) above) with other *abbaco* writings, it becomes obvious that he is conscious of having a particular job to do, and that his work is indeed not only an accidental first extant witness of a new tradition but the very establishment of that tradition: the Tuscan *abbaco* with algebra.[38]

As we know, his treatise is not the first *abbaco*. It is younger than a late thirteenth-century Umbrian specimen drawn from Fibonacci's *Liber abbaci* [ed. Arrighi 1989], which provides us with evidence that an environment had already emerged which felt a need for the new genre. We may presume that Jacopo was deliberately catering for this need, that he deemed insufficient what was already at hand - better, that he "recognized" this insufficiency: the immediate use of his work by others shows that the supposition was justified. This aim is well reflected in the introduction to the treatise, which first states that "el senne è el più nobile thesoro che sia al mondo" and next goes on in the same vein (in high-flown terms that are not borrowed from the scolarly tradition) for a whole page.

In any case he chose to draw on material borrowed from somewhere in order to put together his *Tractatus algorismi*, as he calls it in the *incipit*. The question is, *from where*?

Ultimately, of course, the material comes from the Arabic world. But there is no single Arabism in the work (unless we count *fondaco*, "warehouse", derived from Arabic *funduq*; but this is no mathematical term and will have been naturalized in commercial life before being used in mathematical problems). Jacopo must therefore be assumed to have drawn on a tradition that was already well established in Romance language.

Since the *Algorism* was written in Montpellier, the immediate source can be presumed to be Provence. This is supported by the settings of problems. Rome and Montpellier, together with Florence, Bologna, Avignon, Toulouse, "overseas", Genova, Aigues-Mortes, and Lucca, constitute the horizon of travelling; Florence and Montpellier turn up twice. Other locations, from Paris and Nîmes to Sicily and Alexandria, are only mentioned as domiciles of their currency and measures.

Apart from "overseas", this commercial horizon does not reach the Arabic world. We know, on the other hand, that the Occitan-Catalan area was largely a cultural unity (Paolo Gherardi's commercial horizon encompasses Barcelona as well as Mallorca), and that Catalan trade was mainly directed toward the Arabic world in the second half of the thirteenth century [Abulafia 1985]. The only reasonable assumption seems to be that a tradition for practical reckoning was already established in this Occitan-Catalan orbit around 1300, most likely somewhat earlier, and that Jacopo working (and learning) here decided to spread the gospel. He was not the

38 This being really *first* agrees well with the lack of a standardized terminology which was noticed above. Standardization results from honing and trimming and is difficult to achieve in the first instance (and not very relevant when those for whom texts are produced are not themselves familiar with the standardized terminological canon - in order to make such a public grasp what is meant, variation is more adequate).

only Florentine to go here; in 1328, Paolo Gherardi was to be found in Montpellier - and in the meantime it is a good guess that **V'**, **A"** and **A'** had been produced in the same locality.[39]

This turns a commonly assumed cause-and-effect arrow around: Occitan practical mathematics as known in particular from the fifteenth-century *Algorism* (same name!) from Pamiers[40], and indirectly from Chuquet's *Triparty*, is not primarily derived from Italian *abbaco* mathematics but rather (with ample space for secondary mutual interactions) its root. As Gherardo da Cremona and other scholars would go abroad in the twelfth century in order to get hold of that knowledge for which the schools of their Latin world had discovered an urgent need, so Tuscan masters would go to Montpellier in order to acquire what was needed (for practical use, or culturally) in *their* towns. Whether they stayed like Gherardo "until the end of life" engaged in that quest for knowledge which Jacopo asserted to be more beautiful than anything else or returned like Adelard to pursue a career we shall probably never know.

39 Only linguistic analysis of **M** may tell whether the common precursor of this manuscript and **F** is also likely to have been produced here. The fact that **M** was produced in Genova supports the assumption, but the orthography of **F** (which may of course characterize only **F**) seems very Tuscan.

40 See the partial edition in [Sesiano 1984]. "Algorism" remained the standard name for the type in French and Occitan area - see the *incipit* of Mathieu de Préhoude's treatise as quoted in [Cassinet 1993: 253].

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